AD-A063 252 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 20/11

STABILITY OF A COMPRESSED ROD, (U)

JAN 78 K KREUTZER

UNCLASSIFIED FTD-ID(RS)T-1897-77 NL

1 OF 1

AD A063252

END

DATE FILMED

3-79

DDC

FTD-ID(RS)T-1897-77

1

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

STABILITY OF A COMPRESSED ROD

By

Karl Kreutzer

FOREIGN TECHNOLOGY DIVISION

ACCESSION for		
NTIS	White Section	☒
DDC	Buff Section	☐
UNANNOUNCED		☐
JUSTIFICATION		
BY		
DISTRIBUTION/AVAILABILITY CODES		
Dist.	AVAIL. and/or SPECIAL	
A		

FTD-ID(RS)T-1897-77

EDITED TRANSLATION

FTD-ID(RS)T-1897-77 23 January 1978

MICROFICHE NR: FTD-78-C-000194

STABILITY OF A COMPRESSED ROD

By: Karl Kreutzer

English pages: 39

Source: Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 12, No. 6, December, 1932, pp. 351-368.

Country of origin: East Germany
Translated by: LINGUISTIC SYSTEMS, INC.
F33657-76-D-0389
Aleksandr Kirjanov
Requester: AFFDL/FBR
Approved for public release; distribution unlimited.

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

78 11 22 057

STABILITY OF A COMPRESSED ROD

Karl Kreutzer

Introduction

The elasticity problem of a break [1] is dealt with here from the viewpoint which considers that a break process is inherently a problem of elasticity theory of finite deformations.

It is assumed that the tensions which appear are within the validity field of the expanded Hook law for finite deformations. Included here is the limitation of fixed mass and surface forces, that is independent from deformation.

The method used is an energetic one which takes into account the finiteness of deformations along all three coordinates, as developed by E.Trefftz [2] and reported at the international Congress for Technical Mechanics in Stockholm (1930).

The following nomenclature is used:

$x^{(1)}, x^{(2)}, x^{(3)}$ coordinates of points

X_1, X_2, X_3 components of mass force per volume unit of the non-deformed body along the three directions

Ξ_1, Ξ_2, Ξ_3 components of surface force per area unit of the non-deformed surface along the three directions

$u^{(1)}, u^{(2)}, u^{(3)}$ displacement components in the three directions

$\delta u^{(1)}, \delta u^{(2)}, \delta u^{(3)}$ components of state changes from equilibrium (of disturbances)

E internal energy of the whole body.

An elastic state is a stable equilibrium state when with each finite displacement, compatible with the geometric conditions the increase in internal energy is larger than the work available from external forces. Thus when

$$\Delta E > \iiint \sum_\nu X_\nu \delta u^{(\nu)} dx^{(1)} dx^{(2)} dx^{(3)} + \iint \sum_\nu \Xi_\nu \delta u^{(\nu)} do.$$

Developing both sides according to exponents of $\delta u^{(\nu)}$ and their derivations gives:

$$\Delta E = \delta E + \delta^2 E + \ldots > \iiint \sum_\nu X_\nu \delta u^{(\nu)} dx^{(1)} dx^{(2)} dx^{(3)} + \iint \sum_\nu \Xi_\nu \delta u^{(\nu)} do \quad . \quad . \quad (1).$$

On the right appear no exponents of $\delta u^{(\nu)}$ due to the limitation to fixed external forces X_ν and Ξ_ν.

Should now under any conditions compatible kinematically with $\delta u^{(\nu)}$ the left term be larger than the right one, then the linear terms must disappear. Thus is should be

$$\delta E = \iiint \sum_\nu X_\nu \delta u^{(\nu)} dx^{(1)} dx^{(2)} dx^{(3)} + \iint \sum_\nu \Xi_\nu \delta u^{(\nu)} do.$$

The content of this equation is described as the "principle of virtual displacements". It expresses that for each virtual displacement from equilibrium the change of internal energy equals the work of external forces.

Should the equilibrium state be stable, then the quadratic members of the left side of (1) should outbalance the members on the right side of (1); as a consequence of limiting ourselves to constant external forces, the work of these forces is spent by the linear members. The stability condition is thus reduced to $\delta^2 E > 0$.

The stability limit is reached when for at least one system of displacements $\delta u^{(\nu)}$ the second variation of the internal energy disappears, that is $\delta^2 E = 0$.

These very general points will be, in what follows, applied to the stability problem of the elasticity theory; particularly to the case of obtaining break loads of a compressed rod.

I. Elasticity theory of finite deformations

1.The distorted state

To define the particles of an elastic body "substance coordinates" are used. These are denoted in a non-deformed state as rectangular normal coordinates and indicated by $x^{(\nu)}$ In a distorted state they become curved coordinates corresponding to each mass particle.

Any mass particle before distortion has the coordinates $x^{(1)}, x^{(2)}, x^{(3)}$. It includes a point P towards which, in fixed space intersection lead three vertical vectors $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ - and the location vector $\mathfrak{x} = \sum_\nu x^{(\nu)} \mathfrak{E}_\nu$. A neighboring particle with coordinates $x^{(1)} + dx^{(1)}, x^{(2)} + dx^{(2)}, x^{(3)} + dx^{(3)}$ includes a point Q. If $d\mathfrak{x}$ is a vector of P according to Q then $d\mathfrak{x} = \sum_\nu dx^{(\nu)} \mathfrak{E}_\nu$, and the linear element is

$$ds^2 = \sum_\nu \sum_\mu G_{\nu\mu}\, dx^{(\nu)}\, dx^{(\mu)}.$$

For the coefficients $G_{\nu\mu} = \mathfrak{E}_\nu \cdot \mathfrak{E}_\mu$ of the linear element under normal coordinates the following scheme applies:

$$\| G_{\nu\mu} \| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Through an elastic displacement $\mathfrak{v} = \Sigma u^{(\nu)} \mathfrak{E}_\nu$ the body is distorted. The point $P(\mathfrak{x})$ takes a new position $\bar{P}(\mathfrak{r} = \mathfrak{x} + \mathfrak{v})$ and its neighboring point Q a new position $\bar{Q}(\mathfrak{r} + d\mathfrak{r} = \mathfrak{x} + d\mathfrak{x} + \mathfrak{v} + d\mathfrak{v})$. **Besides,** between the components $\xi^{(\nu)}$ of the location vector $\mathfrak{r}$ after deformation and the components $x^{(\nu)}$ of the location vector $\mathfrak{x}$ before deformation there is a relationship

$$\xi^{(\nu)} = x^{(\nu)} + u^{(\nu)}, \quad d\xi^{(\nu)} = dx^{(\nu)} + du^{(\nu)}.$$

The linear element after distortion is

$$d\sigma^2 = \sum_\nu \sum_\mu \Gamma_{\nu\mu}\, dx^{(\nu)}\, dx^{(\mu)},$$

and where

$$\Gamma_{\nu\mu} = \sum_\varkappa \frac{\partial \xi^{(\varkappa)}}{\partial x^{(\nu)}} \frac{\partial \xi^{(\varkappa)}}{\partial x^{(\mu)}} = G_{\nu\mu} + \frac{\partial u^{(\nu)}}{\partial x^{(\mu)}} + \frac{\partial u^{(\mu)}}{\partial x^{(\nu)}} + \sum_\varkappa \frac{\partial u^{(\varkappa)}}{\partial x^{(\nu)}} \frac{\partial u^{(\varkappa)}}{\partial x^{(\mu)}}$$

The linear element ds^2 has thus been distorted into a linear element $d\sigma^2$. Comparing both changes $\gamma_{\nu\mu} = \Gamma_{\nu\mu} - G_{\nu\mu}$ for the linear elements coefficients. we get:

$$\gamma_{\nu\mu} = \frac{\partial u^{(\nu)}}{\partial x^{(\mu)}} + \frac{\partial u^{(\mu)}}{\partial x^{(\nu)}} + \sum_\varkappa \frac{\partial u^{(\varkappa)}}{\partial x^{(\nu)}} \frac{\partial u^{(\varkappa)}}{\partial x^{(\mu)}}.$$

The $\gamma_{\nu\mu}$ are the "distortion magnitudes" describing the tension and angular changes suffered by each volume element. Due to substitution rules $\gamma_{\nu\mu} = \gamma_{\mu\nu}$ there are only six distinguishable distortion magnitudes. These form a tensor; it corresponds to each point of an elastic body and is its

symmetrical distortion tensor.

The linear element $d\sigma^2$ is expressed in normal rectangular coordinates as:

$$d\sigma^2 = (1+\gamma_{11})\,dx^{(1)2} + (1+\gamma_{22})\,dx^{(2)2} + (1+\gamma_{33})\,dx^{(3)2}$$
$$+ 2\gamma_{12}\,dx^{(1)}\,dx^{(2)} + 2\gamma_{23}\,dx^{(2)}\,dx^{(3)} + 2\gamma_{31}\,dx^{(3)}\,dx^{(1)},$$

where

$$\left.\begin{aligned} \gamma_{11} &= 2\frac{\partial u^{(1)}}{\partial x^{(1)}} + \left(\frac{\partial u^{(1)}}{\partial x^{(1)}}\right)^2 + \left(\frac{\partial u^{(2)}}{\partial x^{(1)}}\right)^2 + \left(\frac{\partial u^{(3)}}{\partial x^{(1)}}\right)^2 \\ \gamma_{12} &= \frac{\partial u^{(1)}}{\partial x^{(2)}} + \frac{\partial u^{(2)}}{\partial x^{(1)}} + \frac{\partial u^{(1)}}{\partial x^{(1)}}\frac{\partial u^{(1)}}{\partial x^{(2)}} + \frac{\partial u^{(2)}}{\partial x^{(1)}}\frac{\partial u^{(2)}}{\partial x^{(2)}} + \frac{\partial u^{(3)}}{\partial x^{(1)}}\frac{\partial u^{(3)}}{\partial x^{(2)}} \end{aligned}\right\} \quad \ldots\ldots (2)$$

etc, through cyclic substitution.

These non-linear equations transform into linear equations of the classical elasticity theory when the products and squares of $\frac{\partial u^{(\nu)}}{\partial x^{(\mu)}}$ may be neglected when compared to linear expressions.

2. Tension state and the equilibrium condition

In order to describe the tension state of a mass particle $(x^{(1)}, x^{(2)}, x^{(3)})$ we regard the square angle paralelopiped which in the non-deformed condition is formed by the elements parallel to the respective axis elements $dx^{(1)}, dx^{(2)}, dx^{(3)}$ from the latter, after deformation a general paralelopiped is formed having edges $\frac{\partial \mathfrak{r}}{\partial x^{(1)}}dx^{(1)}, \frac{\partial \mathfrak{r}}{\partial x^{(2)}}dx^{(2)}, \frac{\partial \mathfrak{r}}{\partial x^{(3)}}dx^{(3)}$.
The vectors $\mathfrak{e}_\mu = \frac{\partial \mathfrak{r}}{\partial x^{(\mu)}}$, which give the direction and the increase ratio after deformation, are called "grid vectors".

When the force $\mathfrak{f}^{(1)}\,dx^{(2)}\,dx^{(3)}$ acts in the direction of growing $x^{(1)}$ boundary surface of the parallelopiped

, then $\mathfrak{t}^{(1)}$ is called the tension vector for surfaces $x^{(1)}$ = constant. The same corresponds to the remaining surfaces. $\mathfrak{t}^{(\nu)}$ is the tension vector for the surface element $x^{(\nu)}$=constant and signifies a force per unit of the non-deformed surface.

Each of the three tension vectors may be decomposed according to the grid vectors

$$\mathfrak{t}^{(\nu)} = \sum_{\nu\mu} k^{\nu\mu} \mathfrak{e}_\mu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3),$$

by which nine tension components $k^{\nu\mu}$ are created, which completely describe the tension state.

The equilibrium against torsion in any chosen direction demands the disappearance of the momentums' sum of all the forces acting upon the paralelopiped considered at its center, thus:

$$\Sigma M = \sum_\nu (\mathfrak{e}_\nu \times \mathfrak{t}^{(\nu)})\, dx^{(1)}\, dx^{(2)}\, dx^{(3)} = 0.$$

After introducing the reciprocal vectors $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$ of grid vectors $\mathfrak{e}^1, \mathfrak{e}^2, \mathfrak{e}^3$ according to formulas [3]

$$\mathfrak{e}^{(1)} = \frac{\mathfrak{e}_2 \times \mathfrak{e}_3}{[\mathfrak{e}_1\, \mathfrak{e}_2\, \mathfrak{e}_3]} \text{ and } \mathfrak{e}^{(\nu)} \cdot \mathfrak{e}_\mu = \begin{matrix} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{matrix}$$

and keeping in mind (3)

$$\mathfrak{e}^{(1)}(k^{23} - k^{32}) + \mathfrak{e}^{(2)}(k^{31} - k^{13}) + \mathfrak{e}^{(3)}(k^{12} - k^{21}) = 0.$$

This vectorial equation is valid only when,

$$k^{\nu\mu} = k^{\mu\nu}.$$

Following this Cauchy reciprocity law is valid for $k^{\nu\mu}$. The number of tension magnitudes is reduced from nine to six.

For the equilibrium against a displacement in any direction it is necessary to consider the action of force $\mathfrak{t}^{(1)}\, dx^{(2)}\, dx^{(3)}$ on the paralelopipeds' surface element $x^{(1)}$=constant on to the surface element $x^{(1)} + dx^{(1)}$ = constant; the acting force is then $\mathfrak{t}^{(1)}\, dx^{(2)}\, dx^{(3)} + \frac{\partial \mathfrak{t}^{(1)}}{\partial x^{(1)}}\, dx^{(1)}\, dx^{(2)}\, dx^{(3)}$.

The force excess amounts for this surface pair $\frac{\partial \mathfrak{t}^{(1)}}{\partial x^{(1)}} dx^{(2)} dx^{(3)} dx^{(1)}$.
Through cyclical substitution follow the force excesses for the remaining surface pairs. A resulting $d\mathfrak{R}$ of the tension forces is obtained which is

$$d\mathfrak{R} = \sum_{\nu} \frac{\partial \mathfrak{t}^{(\nu)}}{\partial x^{(\nu)}} dx^{(1)} dx^{(2)} dx^{(3)}.$$

Besides a mass force acts upon the paralelopiped

$$d\mathfrak{K} = \mathfrak{P} dx^{(1)} dx^{(2)} dx^{(3)} = \sum_{\nu} p^{(\nu)} \mathfrak{e}_{\nu} dx^{(1)} dx^{(2)} dx^{(3)},$$

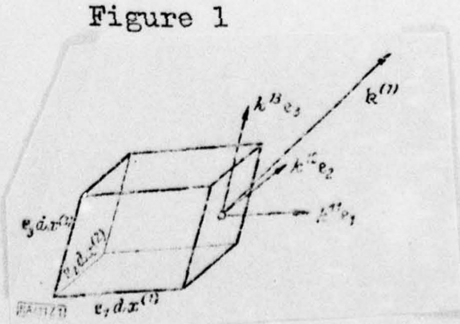
where $\mathfrak{P}$ is the mass force per volume unit of the non--deformed body and $p^{(\nu)}$ its component in the direction of the th grid vector. The equilibrium against displacement requires

$$d\mathfrak{R} + d\mathfrak{K} = 0,$$

from here after abbreviating through the product $dx^{(1)} dx^{(2)} dx^{(3)}$

$$\sum_{\nu} \frac{\partial \mathfrak{t}^{(\nu)}}{\partial x^{(\nu)}} + \sum_{\nu} p^{(\nu)} \mathfrak{e}_{\nu} = 0.$$

Figure 1



Using formula (3) and differentiating it follows:

$$\sum_\nu \sum_\mu \frac{\partial k^{\nu\mu}}{\partial x^{(\nu)}} c_\mu + \sum_\nu \sum_\mu k^{\nu\mu} \frac{\partial c_\mu}{\partial x^{(\nu)}} + \sum_\nu p^{(\nu)} c_\nu = 0.$$

Thus it is

$$\frac{\partial c_\mu}{\partial x^{(\nu)}} = \sum_h \begin{Bmatrix} \mu\nu \\ h \end{Bmatrix} c_h.$$

where $\begin{Bmatrix} \mu\nu \\ h \end{Bmatrix}$ are Christoffer's symbols of second class.

At the same time:

$$\sum_\nu \sum_\mu \frac{\partial k^{\nu\mu}}{\partial x^{(\nu)}} c_\mu + \sum_\nu \sum_\mu \sum_h k^{\nu\mu} \begin{Bmatrix} \mu\nu \\ h \end{Bmatrix} c_h + \sum_\nu p^{(\nu)} c_\nu = 0.$$

Should the base vector be everywhere called c_h, an index exchange has to be effected. If we decompose this vectorial equation into its components we get

$$\sum_\nu \frac{\partial k^{\nu h}}{\partial x^{(\nu)}} + \sum_\nu \sum_h k^{\nu\mu} \begin{Bmatrix} \mu\nu \\ h \end{Bmatrix} + p^{(h)} = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

This equation delivers for each of the three directions $h=1,2,3$ a partial differential equation. The middle member considers the bending of the coordinate curves, however the diferentiation in (4) follows the substance coordinates which are curved in the deformed state.

3. Internal energy

The approach up to now gives each point a distortion and tension state. To represent the interconnection of both we have to consider the internal energy. Its existence follows for all reversible static phenomena from thermodynamic considerations.

The following meanings are used:

$\varepsilon = [\mathfrak{e}_1 \mathfrak{e}_2 \mathfrak{e}_3] = \sqrt{\lvert \Gamma_{\mu\nu} \rvert}$	volume of each grid unit
$dV = \varepsilon\, dx^{(1)}\, dx^{(2)}\, dx^{(3)}$	volume of an infinitesimal paralelopiped
e	Internal energy per Volume unit - non deformed body
$e\varepsilon$	internal energy of each grid unit
$d^3E = e\,dV$	internal energy of volume dV

In order to represent the internal energy e as a function of the distortion magnitudes $\gamma_{\nu\mu}$, we have to find the invariants of the distorted state. An absolute invariant for all values of paramater λ is

$$\frac{1}{\lvert G_{\nu\mu} \rvert} \lvert \gamma_{\nu\mu} - \lambda G_{\nu\mu} \rvert .$$

By developing the determinants by exponents of λ a 3d degree function in λ results. Since this is an invariant for all λ the coefficients of the cubic form must also be invariants. Thus three invariants are formed:

$$
\begin{aligned}
I_1 = \frac{1}{\lvert G_{\nu\mu} \rvert} \{ & \gamma_{11}(G_{22}G_{33} - G_{23}{}^2) + \gamma_{22}(G_{33}G_{11} - G_{31}{}^2) + \gamma_{33}(G_{11}G_{22} - G_{12}{}^2) \\
& + 2\gamma_{12}(G_{31}G_{23} - G_{12}G_{33}) + 2\gamma_{23}(G_{12}G_{13} - G_{23}G_{11}) + 2\gamma_{31}(G_{23}G_{12} - G_{31}G_{22}) \}, \\
I_2 = \frac{1}{\lvert G_{\nu\mu} \rvert} \{ & G_{11}(\gamma_{22}\gamma_{33} - \gamma_{23}{}^2) + G_{22}(\gamma_{33}\gamma_{11} - \gamma_{31}{}^2) + G_{33}(\gamma_{11}\gamma_{22} - \gamma_{12}{}^2) \\
& + 2G_{12}(\gamma_{31}\gamma_{23} - \gamma_{12}\gamma_{33}) + 2G_{23}(\gamma_{12}\gamma_{13} - \gamma_{23}\gamma_{11}) + 2G_{31}(\gamma_{23}\gamma_{12} - \gamma_{31}\gamma_{22}) \}, \\
I_3 = \frac{\lvert \gamma_{\nu\mu} \rvert}{\lvert G_{\nu\mu} \rvert} & .
\end{aligned}
$$

For $\gamma_{\nu\mu}$ I_1 is linear, I_2 is quadratic and I_3 cubical. further, I_2 follows from I_1 through a substitution of G with γ.

For rectangular normal coordinates it is:

$$I_1 = \gamma_{11} + \gamma_{22} + \gamma_{33},$$
$$I_2 = \gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{33} + \gamma_{33}\gamma_{11} - \gamma_{12}^2 - \gamma_{23}^2 - \gamma_{31}^2,$$
$$I_3 = |\gamma_{\nu\mu}|.$$

What is to be now attached for e ? From numerous possibilities we take the simplest which corresponds to the classical theory. Thus εe is represented by the simplest quadratic invariant of the distorted state:

$$e = \frac{\sqrt{|G_{\nu\mu}|}}{\sqrt{|\Gamma_{\nu\mu}|}} \left\{ \frac{\alpha}{2} I_1^2 - \beta I_2 \right\}.$$

The forward factor is as a volume ratio of the paralelopiped between the non-deformed and deformed states - an invariant. The constants α and β are the two independent elasticity constants of the classical theory

$$\alpha = \frac{G}{2} \frac{m-1}{m-2}, \quad \beta = \frac{G}{2}.$$

G means here the thrust modulus and m Poisson's number of the cross-sectional contraction of the material.

The internal energy for each grid unit is thus

$$\varepsilon e = \sqrt{|G_{\nu\mu}|} \left\{ \frac{\alpha}{2} I_1^2 - \beta I_2 \right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (5).$$

From the condition that e has to be positive it follows just as in the classical elasticity theory $2 \leqq m \leqq \infty$. $2 \leqq m \leqq \infty$.

For this span (5) is applicable, since (εe) is a positive, definite, quadratic homogeneous form in $\gamma_{\nu\mu}$ $\gamma_{\nu\mu}$.

For rectangular normal coordinates it is

$$e\varepsilon = \frac{a}{2}(\gamma_{11}+\gamma_{22}+\gamma_{33})^2 - \beta(\gamma_{11}\gamma_{22}+\gamma_{22}\gamma_{33}+\gamma_{33}\gamma_{11}-\gamma_{12}^2-\gamma_{23}^2-\gamma_{31}^2).$$

The infinitely small deformations may easily be arranged into (5), when the linear expressions are introduced for the distortion magnitudes.

4. Tension-expansion equations and Hooke's law extended to finite deformations

The equilibrium conditions (4) are not sufficient alone to determine all tension and distortion magnitudes. For this we also need the relationships between the acting forces and the deformations caused by them.

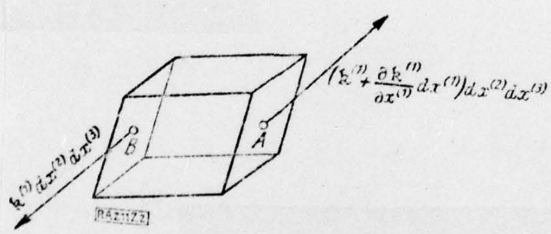


Figure 2

As in the classical theory a possibility is available when observing the internal energy, to deduce tension-expansion equations, if we part from the internal energy of each grid unit.

We will take a real displacement v and add to it an additional displacement δv and the $\gamma_{\nu\mu}$ will change by $\delta\gamma_{\nu\mu}$. The increase in internal energy is equal to the work exerted by the tension forces upon a volume element.

Point A suffers an additional displacement $\delta\mathfrak{v} + \frac{\partial\delta\mathfrak{v}}{\partial x^{(1)}}\frac{dx^{(1)}}{2}$ and point B $\delta\mathfrak{v} - \frac{\partial\delta\mathfrak{v}}{\partial x^{(1)}}\frac{dx^{(1)}}{2}$. With this additional displacement an internal energy change is obtained $\mathfrak{k}^{(1)} \cdot \frac{\partial\delta\mathfrak{v}}{\partial x^{(1)}} dx^{(1)} dx^{(2)} dx^{(3)}$ Through cyclical substitutions internal energy changes are obtained for the remaining two directions. Their sum comprises per grid unit:

$$\delta(e\varepsilon) = \sum_{\nu} \mathfrak{k}^{(\nu)} \cdot \frac{\partial\delta\mathfrak{v}}{\partial x^{(\nu)}} = \sum_{\nu}\sum_{\mu} k^{\nu\mu}\, \mathfrak{e}_{\mu} \cdot \frac{\partial\delta\mathfrak{v}}{\partial x^{(\nu)}}$$

now $\mathfrak{e}_{\nu} = \mathfrak{E}_{\nu} + \frac{\partial\mathfrak{v}}{\partial x^{(\nu)}}$ **with** $\frac{\partial\delta\mathfrak{v}}{\partial x^{(\nu)}}$; $\Gamma_{\nu\mu} = G_{\nu\mu} + \gamma_{\nu\mu}$, **with** $\delta\Gamma_{\nu\mu} = \delta\gamma_{\nu\mu}$,

and because $\Gamma_{\nu\mu} = \mathfrak{e}_{\nu} \cdot \mathfrak{e}_{\mu}$ it is $\delta\Gamma_{\nu\mu} = \mathfrak{e}_{\nu} \cdot \delta\mathfrak{e}_{\mu} + \mathfrak{e}_{\mu} \cdot \delta\mathfrak{e}_{\nu}$

The following representations are obtained for the internal energy change:

$$\delta(e\varepsilon) = k^{11}\frac{\delta\gamma_{11}}{2} + k^{22}\frac{\delta\gamma_{22}}{2} + k^{33}\frac{\delta\gamma_{33}}{2} + k^{12}\,\delta\gamma_{12} + k^{23}\,\delta\gamma_{23} + k^{31}\,\delta\gamma_{31} \quad \ldots \quad (6)$$

On the other hand based on formula (5)

$$\delta(e\varepsilon) = \frac{\partial(e\varepsilon)}{\partial\gamma_{11}}\delta\gamma_{11} + \frac{\partial(e\varepsilon)}{\partial\gamma_{22}}\delta\gamma_{22} + \frac{\partial(e\varepsilon)}{\partial\gamma_{33}}\delta\gamma_{33} + \frac{\partial(e\varepsilon)}{\partial\gamma_{12}}\delta\gamma_{12} + \frac{\partial(e\varepsilon)}{\partial\gamma_{23}}\delta\gamma_{23} + \frac{\partial(e\varepsilon)}{\partial\gamma_{31}}\delta\gamma_{31} \quad \ldots \quad (6a)$$

Comparing both expressions we get the six equations

$$\left.\begin{array}{ll} k^{11} = \dfrac{2\,\partial(e\varepsilon)}{\partial\gamma_{11}}, & k^{12} = \dfrac{\partial(e\varepsilon)}{\partial\gamma_{12}}, \\ k^{22} = \dfrac{2\,\partial(e\varepsilon)}{\partial\gamma_{22}}, & k^{23} = \dfrac{\partial(e\varepsilon)}{\partial\gamma_{23}}, \\ k^{33} = \dfrac{2\,\partial(e\varepsilon)}{\partial\gamma_{33}}, & k^{31} = \dfrac{\partial(e\varepsilon)}{\partial\gamma_{31}} \end{array}\right\} \quad \ldots\ldots\ldots \quad (7).$$

These are the tension-expansion equations of the finite defor-

mations theory. They express that the tension magnitudes $k^{\nu\mu}$ may be gotten through a partial differentiation of specific internal energy $(e\epsilon)$ according to the distortion magnitudes $\gamma_{\nu\mu}$. They are valid in form (7) even when not starting from normal coordinates.

According to (5) the internal energy $(e\epsilon)$ is a homogeneous quadratic form of the distortion magnitudes $\gamma_{\nu\mu}$. The $k^{\nu\mu}$ are according to (7) linear homogeneous functions of the distortion magnitudes and are:

$$\left.\begin{aligned} k^{11} &= \frac{2}{\sqrt{|G_{\nu\mu}|}}\left[\alpha I_1 (G_{22} G_{33} - G_{23}^2) - \beta (G_{22} \gamma_{33} + G_{33} \gamma_{11} - 2 G_{23} \gamma_{23})\right], \\ k^{12} &= \frac{2}{\sqrt{|G_{\nu\mu}|}}\left[\alpha I_1 (G_{13} G_{23} - G_{12} G_{33}) - \beta (G_{23} \gamma_{13} + G_{13} \gamma_{23} - G_{33} \gamma_{12} - G_{12} \gamma_{33})\right] \end{aligned}\right\} \quad . \; . \quad (8)$$

etc, through cyclical substitutions.

These equations are Hook's law extended to finite deformations. For rectangular normal coordinates it is

$$\left.\begin{aligned} k^{11} &= G\left(\gamma_{11} + \frac{\Phi}{m-2}\right), & k^{12} &= G \gamma_{12}, \\ k^{22} &= G\left(\gamma_{22} + \frac{\Phi}{m-2}\right), & k^{23} &= G \gamma_{23}, \\ k^{33} &= G\left(\gamma_{33} + \frac{\Phi}{m-2}\right), & k^{31} &= G \gamma_{31} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9).$$

where $\Phi = \gamma_{11} + \gamma_{22} + \gamma_{33}$ ist.

Equations (9) convert into the classical tension-expansion equations when infinitely small deformations are allowed.

Then with

$$\frac{\partial u^{(1)}}{\partial x^{(1)}} = u_x, \qquad \frac{\partial u^{(2)}}{\partial x^{(2)}} = v_y, \qquad \frac{\partial u^{(3)}}{\partial x^{(3)}} = w_z,$$

$$\Phi = 2(u_x + v_y + w_z) = 2\,\Theta,$$

$$k^{11} = 2\,G\left(u_x + \frac{\Theta}{m-2}\right), \qquad k^{12} = G\,(u_y + v_x),$$

$$k^{22} = 2\,G\left(v_y + \frac{\Theta}{m-2}\right), \qquad k^{23} = G\,(v_z + w_y),$$

$$k^{33} = 2\,G\left(w_z + \frac{\Theta}{m-2}\right), \qquad k^{31} = G\,(w_x + u_z).$$

Equations (9) supply the connection between the tension components and the distortion magnitudes. If the tension component's $k^{\nu\mu}$ values from (9) are put into the equilibrium conditions (4) we obtain for the determination of the equilibrium the necessary differential equations for the displacement components $u^{(1)}, u^{(2)}, u^{(3)}$. For these differential equations (exactly as in the classical elasticity theory) correspond limiting conditions which ma refer to displacements or to surface forces.

5. Stability of equilibrium

Above we presented equations which serve to ascertain he equilibrium. For the study of stability it is anticipated that these equations be integrated so that we may know the tensions and displacements of the equilibrium studied for stability. To apply the theory we sha l satisfy ourselves with a relatively simple particular case - a compressed rod- whose tension and displacement states may simply be overlooked.

To deal with the stability we have to compare the internal energy of the body in equilibrium state with the internal energy in a neighboring state. Thus, besides considering the equilibrium displacements $u^{(1)}, u^{(2)}, u^{(3)}$ we have to consider the neighboring displacements $\{u^{(1)} + \delta u^{(1)},\ u^{(2)} + \delta u^{(2)},\ u^{(3)} + \delta u^{(3)}\}$ and the expression for the internal energy becomes

$$E = \iiint (c\,\varepsilon)\, d x^{(1)}\, d x^{(2)}\, d x^{(3)}$$

in the periphery of the equilibrium state by exponents of $\delta u^{(3)}$ and eventually developed by their derivation.

According to the statements in the introduction the equilibrium will be stable when for each allowable (thus compatible with the geometric conditions) system of "disturbances" $\delta u^{(2)}$ the so called second variation $\delta^2 E$ (which when developed contains the quadratic members) is always positive.

If we describe the differentiations by the following indices, for exampel $u_i^{(r)} = \frac{\partial u^{(r)}}{\partial x^{(i)}}$, then the second variation will acquire the following form 4)

$$\begin{aligned}
\delta^2 E = \iiint \Big\{ & \frac{1}{2} \sum_{h} \left[4\alpha\left(1+u_h^{(h)}\right)^2 + 2\beta\left(u_{h+1}^{(h)}\right)^2 + 2\beta\left(u_{h+2}^{(h)}\right)^2 + k^{hh}\right] \delta u_h^{(h)}\, \delta u_h^{(h)} \\
& + \frac{1}{2} \sum_{h,k} \left[2\beta\left(1+u_h^{(h)}\right)^2 + 2\beta\left(u_{2(h+k)}^{(h)}\right)^2 + 4\alpha\left(u_k^{(h)}\right)^2 + k^{kk}\right] \delta u_k^{(h)}\, \delta u_k^{(h)} \\
& + \sum_{h,k} \left[2(\alpha-\beta)\left(1+u_h^{(h)}\right) u_k^{(h)} + k^{hk}\right] \delta u_h^{(h)}\, \delta u_k^{(h)} \\
& + \sum_{h,k} \left[4\alpha\left(1+u_h^{(h)}\right) u_h^{(k)} + 2\beta\left(1+u_k^{(k)}\right) u_k^{(h)} + 2\beta u_{2(h+k)}^{(h)}\, u_{2(h+k)}^{(h)}\right] \delta u_h^{(h)}\, \delta u_h^{(k)} \\
& + \sum_{h,k} \left[4(\alpha-\beta)\left(1+u_h^{(h)}\right)\left(1+u_k^{(k)}\right) + 2\beta u_k^{(h)}\, u_h^{(k)}\right] \delta u_h^{(h)}\, \delta u_k^{(k)} \\
& + \sum_{k,k,m}' \left[4(\alpha-\beta)\left(1+u_h^{(h)}\right) u_m^{(k)} + 2\beta u_h^{(h)}\, u_m^{(h)}\right] \delta u_h^{(h)}\, \delta u_m^{(k)} \\
& + \sum_{h,k,m}' \left[2(2\alpha-\beta)\, u_m^{(h)}\, u_k^{(h)} + k^{mk}\right] \delta u_m^{(h)}\, \delta u_k^{(h)} \\
& + \sum_{h,k,m}' \left[2\beta\left(1+u_h^{(h)}\right)\left(1+u_k^{(k)}\right) + 4(\alpha-\beta)\, u_h^{(k)}\, u_k^{(h)}\right] \delta u_k^{(h)}\, \delta u_h^{(k)} \\
& + \sum_{h,k,m} \left[4(\alpha-\beta)\, u_m^{(h)}\, u_h^{(k)} + 2\beta\left(1+u_k^{(h)}\right) u_m^{(k)}\right] \delta u_h^{(k)}\, \delta u_m^{(k)} \\
& + \sum_{n,k,m}' \left[2\beta\left(1+u_h^{(h)}\right) u_h^{(k)} + 4\alpha u_k^{(m)}\, u_h^{(k)} + 2\beta\left(1+u_k^{(k)}\right) u_k^{(h)}\right] \delta u_h^{(k)}\, \delta u_k^{(m)} \Big\}\, d x^{(1)}\, d x^{(2)}\, d x^{(3)}
\end{aligned} \qquad (10).$$

The indices go through values from one to three and two differently numbered indices may never acquire the same value. Values above three of the indices are to be reduced by modulus three.

Now the matter is to decide in special cases if such allowable disturbances $\delta u^{(\nu)}$ may be found for which the second variations becomes negative or if it remains positive. Now then the displacements and tensions of the equilibrium state are to be regarded as given and we should now look for the "most dangerous" disturbances, such $\delta u^{(\nu)}$ which would possibly allow the second variation to be negative.

6. Determination of the stability limit

To determine the stability limit of an elastic equilibrium state we need to consider the second variation of the internal energy. It may be obtained from expression (10) if we introduce into it the values corresponding to this special case for displacement and tension components. Since the quadratic form in $\delta u^{(\nu)}$ under the integral and its derivations may always be written as a difference of two positively defined forms, we should write:

$$\delta^2 E = Q_1 - Q_2 ,$$

Where Q_1 and Q_2 are integrals over positively definite quadratic forms in the derivations of $\delta u^{(\nu)}$.

To find if the difference could become negative, we write:

$$\delta^2 E = Q_2 (\lambda - 1) , \qquad \lambda = \frac{Q_1}{Q_2} .$$

The smaller λ becomes the higher the danger for $\delta_2 E$ to become

negative. The "most dangerous" displacement from equilibrium state is such for which λ becomes a minimum.

If λ becomes a minimum then $\delta\lambda = \frac{Q_2 \delta Q_1 - Q_1 \delta Q_2}{Q_2^2}$ must disappear, thus it should be

$$\delta Q_1 = \lambda \delta Q_2 \quad \ldots \quad (11)$$

for all allowable variations $\delta(\delta u^{(\nu)})$.

Equation (11) has a solution $\lambda = \bar{\lambda}$ which may be larger, equal or smaller than one. The case $\bar{\lambda} = 1$ gives the stability limit.

Since we are asking only for a stability limit, the problem may be simplified further. Equation (11) says: for the "most dangerous" displacement $\delta u^{(1)}, \delta u^{(2)}, \delta u^{(3)}$ from equilibrium state is for all changes $\delta(\delta u^{(\nu)})$

$$\delta Q_1 = \lambda \delta Q_2.$$

Now, instead of finding the value λ for all the given forces and then ask for which forces $\lambda = 1$, we may directly put $\lambda = 1$. Then we will seek the forces and the corresponding displacements $\delta u^{(1)}, \delta u^{(2)}, \delta u^{(3)}$ from equilibrium state; for any $\delta(\delta u^{(\nu)})$ it will be

$$\delta Q_1 = \delta Q_2 \quad \ldots \quad (12)$$

This equation is actually Jacob's criterion for the access of stability change. It allows the obtainment of the stability limit for any equilibrium state.

It should be noted that the stability limit may also be determined from the isoperimetric problem: among all the allowable displacement variations from the equilibrium state the "most dangerous" are those for which the integral Q_1

becomes a minimum for a neighboring condition $Q_2=1$. With λ as the Lagrange factor for the neighboring condition it becomes identical to equation

$$\delta Q_1=\lambda\delta Q_2$$

which agrees with (11).

The stability limit criterion expressed here in connection with the variation calculation may also be converted into a homogeneous system of differential equations and homogeneous limiting conditions for the "most dangerous" variation. Such a homogeneous problem has several solutions different from zero when the parameters characterizing the equilibrium state (loads) acquire definite critical values (break values).

This method will be used below with the example of a compressed rod.

II. Stability of a Compressed Rod

7. Equilibrium state

To carry out the Trefftz method in order to find the stability limit of a compressed rod the definitions should be fitted to this particular problem.

The coordinates are designated x,y,z. The coordinate system is layed out so that the z-axis coincides with the rod axis and the remaining axis fall within the cross-section of the rod.

The equilibrium displacements are indicated with capital letters U,V,W. Variations of U,V,W will be indicated by small u,v,w and the variations of these latter will be indicated by δu, δv, δw .

The rod considered should be "fixed". This means that all points of the lower and all points of the uppper cross-sections

have suffered the same vertical displacement; thus W should be predetermined for the cross-sections of the extremes. From all the disturbances of the equilib ium state only those are acceptable whose w=0 at the frontal surfaces of the rod, and thus also

$\frac{\delta w}{\delta x} = 0$ and $\frac{\delta w}{\delta y} = 0$.

The rod is held by its lower end (z=0) and compressed by a portion ΔL by a force P=p•f (f is the cross-section). then the displacements are

$$U = a_x x; \qquad V = a_y y; \qquad W = a_z z,$$

and these may be neglected.

The connection between the expansions a_x, a_y, a_z and pressure p is obtained from the tension-expansion equations (9) which now appear as

$$\kappa^{xx} = G\left(\gamma_{xx} + \frac{\Phi}{m-2}\right) = 0,$$

$$k^{yy} = G\left(\gamma_{yy} + \frac{\Phi}{m-2}\right) = 0,$$

$$k^{zz} = G\left(\gamma_{zz} + \frac{\Phi}{m-2}\right) = -p.$$

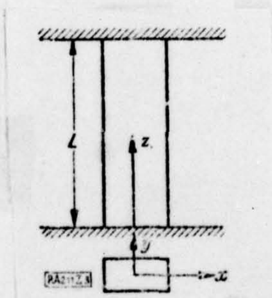


Figure 3

it is

$$\Phi = -\frac{m-2}{m+1}\frac{p}{G},$$

$$\gamma_{xx} = \gamma_{yy} = \frac{p}{(m+1)\,G}, \qquad \gamma_{zz} = -\frac{m\,p}{(m+1)\,G},$$

and

$$
\begin{aligned}
\gamma_{xx} &= 2\,a_x + a_x^2\,, & a_x &= \sqrt{1+\gamma_{xx}} - 1\,,\\
\gamma_{yy} &= 2\,a_y + a_y^2\,, & a_y &= \sqrt{1+\gamma_{yy}} - 1\,,\\
\gamma_{zz} &= 2\,a_z^2 + a_z^2\,, & a_z &= \sqrt{1+\gamma_{zz}} - 1\,.
\end{aligned}
$$

The equilibrium conditions (4) are (as in any tension state) self-fulfilling.

8. Second variation of the internal energy

The homogeneous equilibrium loses above a "critical" magnitude of pressure p its stability.

To find the stability limit the second internal energy variation has to be considered. It results in :

$$
\begin{aligned}
\delta^2 E = \iiint \{ & 2\,\alpha\,[(1+a_x)^2\,u_x^2 + (1+a_y)^2\,v_y^2 + (1+a_z)^2\,w_z^2]\\
& + 4\,(\alpha-\beta)\,[(1+a_x)(1+a_y)\,u_x\,v_y + (1+a_y)(1+a_z)\,v_y\,w_z + (1+a_z)(1+a_x)\,w_z\,u_x\\
& + \beta\,[(1+a_x)^2\,(u_y^2+u_z^2) + (1+a_y)^2\,(v_x^2+v_z^2) + (1+a_z)^2\,(w_x^2+w_y^2)\\
& + 2\,[(1+a_x)(1+a_y)\,v_x\,u_y + 2\,(1+a_y)(1+a_z)\,w_y\,v_z + 2\,(1+a_z)(1+a_x)\,u_z\,w_x]\}\,dx\,dy\,dz\\
& \frac{p}{2}\iiint (u_z^2 + v_z^2 + w_z^2)\,dx\,dy\,dz\,.
\end{aligned}
$$

introducing instead of disturbances u,v,w

$$\bar{u} = (1+a_x)\,u\,,\qquad \bar{v} = (1+a_y)\,v\,,\qquad \bar{w} = (1+a_z)\,w$$

and with $\alpha = \frac{G\,m}{2\,m-2}$, $\beta = \frac{G}{2}$ it is

$$\delta^2 E = \iiint \left\{ \frac{G}{2}\frac{m}{m-2}(\bar u_x + \bar v_y + \bar w_z)^2 - 2G\left[\bar u_x \bar v_y + \bar v_y \bar w_z + \bar w_z \bar u_x - \frac{1}{4}(\bar u_y + \bar v_x)^2 - \frac{1}{4}(\bar v_z + \bar w_y)^2 - \frac{1}{4}(\bar w_x + \bar u_z)^2 \right] \right\} dx\,dy\,dz$$

$$- \frac{p}{2} \iiint \left[\frac{\bar u_z^2}{(1+a_x)^2} + \frac{\bar v_z^2}{(1+a_y)^2} + \frac{\bar w_z^2}{(1+a_z)^2} \right] dx\,dy\,dz.$$

The first integral is equal exactly to the form change work of the classical elasticity theory. It shows the form modification work involved in small displacements $\bar u, \bar v, \bar w$ when these alone are present. The second integral contains the acting pressure p. And so the second variation of the internal energy of a compressed rod is written as a difference of two integrals over a positively defined quadratic forms in derivations of $\bar u, \bar v, \bar w$,

thus:

$$\delta^2 E = Q_1 - Q_2$$

9.Jacobi's equations

After developments in 6, we reach the stability limit for an elastic equilibrium state; if there is a "most dangerous" variation u,v,w for which with every acceptable variation $\delta u, \delta v, \delta w$ it is

$$\delta Q_1 = \delta Q_2$$

and for the case of a compressed rod Q_1 and Q_2 have the above values.

The satisfaction of equation (12) for any variations of u,v,w leads to a system of three partial differential equations. It represents Jacobi's equations adapted to an equilibrium problem in a form of a variation's problem (6).

To simplify the derivation the crosslines above $\bar u, \bar v, \bar w$ are again omitted. The partial derivatives of the integrants of Q_1 along the break magnitudes $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ etc are in formal accordance with the classical theory designated by the tensions σ_x, τ_{xy} etc.

$$\sigma_x = 2G\left\{\frac{\partial u}{\partial x} + \frac{\Theta}{m-2}\right\} \quad \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}$$ etc. where $$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Since Q_1 has the shape of Hook's form change work, the first variation of Q_1 results as when $\Theta = u_x + v_y + w_z$

$$\begin{aligned}
\delta Q_1 = {} & 2\iiint \delta u\, G\left(\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right) dx\,dy\,dz \\
& + 2\iiint \delta v\, G\left(\Delta v + \frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right) dx\,dy\,dz \\
& + 2\iiint \delta w\, G\left(\Delta w + \frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right) dx\,dy\,dz \\
& - 2\iint_{\text{Mantel}} \{\delta u\,[\sigma_x \cos(n,x) + \tau_{xy}\cos(n,y)] + \delta v\,[\tau_{xy}\cos(n,x) + \sigma_y \cos(n,y)] \\
& + \delta w\,[\tau_{zx}\cos(n,x) + \tau_{zy}\cos(n,y)]\}\, do \\
& - \iiint_{\text{ob. Stirnfl.}} [\delta u\,\tau_{zx} + \delta v\,\tau_{zy} + \delta w\,\sigma_z]\, do + \iiint_{\text{unt. Stirnfl.}} [\delta u\,\tau_{zx} + \delta v\,\tau_{zy} + \delta w\,\sigma_z]\, do.
\end{aligned}$$

shell

upper front surface

lower front surface

Further

$$\begin{aligned}
\delta Q_2 = {} & \frac{2p}{(1+\alpha_x)^2}\iiint \delta u \frac{\partial^2 u}{\partial z^2} dx\,dy\,dz + \frac{2p}{(1+\alpha_y)^2}\iiint \delta v \frac{\partial^2 v}{\partial z^2} dx\,dy\,dz \\
& + \frac{2p}{(1+\alpha_z)^2}\iiint \delta w \frac{\partial^2 w}{\partial z^2} dx\,dy\,dz \\
& - \iiint_{\text{ob. Stirnfl.}} \left| \frac{p}{(1+\alpha_x)^2}\frac{\partial u}{\partial z}\delta u + \frac{p}{(1+\alpha_y)^2}\frac{\partial v}{\partial z}\delta v + \frac{p}{(1+\alpha_z)^2}\frac{\partial w}{\partial z}\delta w \right| do \\
& + \iiint_{\text{unt. Stirnfl.}} \left| \frac{p}{(1+\alpha_x)^2}\frac{\partial u}{\partial z}\delta u + \frac{p}{(1+\alpha_y)^2}\frac{\partial v}{\partial z}\delta v + \frac{p}{(1+\alpha_z)^2}\frac{\partial w}{\partial z}\delta w \right| do.
\end{aligned}$$

upper front surface

lower front surface

Equation $\delta Q_1 = \delta Q_2$ may be possible for all acceptable variations only when the integrants in the space integrals are mutually equal.

It follows:

$$
\left.\begin{aligned}
G\left(\Delta u+\frac{m}{m-2}\frac{\partial\Theta}{\partial x}\right)&=\frac{p}{(1+a_x)^2}\frac{\partial^2 u}{\partial z^2},\\
G\left(\Delta v+\frac{m}{m-2}\frac{\partial\Theta}{\partial y}\right)&=\frac{p}{(1+a_y)^2}\frac{\partial^2 v}{\partial z^2},\\
G\left(\Delta w+\frac{m}{m-2}\frac{\partial\Theta}{\partial z}\right)&=\frac{p}{(1+a_z)^2}\frac{\partial^2 w}{\partial z^2}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(13).
$$

These are Jacobi's equations.

Limiting conditions belong to the above and follow from the surface integrals.

a) Integrals touched only by δQ_1 over the shell plane have to dissappear. It follows from this that on the shell surface following equations apply:

$$
\begin{aligned}
\sigma_x\cos(n,x)+\tau_{xy}\cos(n,y)&=0\\
\tau_{xy}\cos(n,x)+\sigma_y\cos(n,y)&=0\qquad\qquad(\cos n,z=0)\\
\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)&=0
\end{aligned}
$$

They indicate that a force-free shell plane corresponds to the displacements u,v,w.

b) The integrals of δQ_1 and δQ_2 taken over the front surfaces must be equal. Since according to 7, for all allowable disturbances and their variations at both front surfaces are w=0 and $\delta w=0$ for all points, also $w_x=0$ and w=0, it follows that for the front surfaces $\tau_{zx}=Gu_z$ and $\tau_{xy}=Gv_z$. So for the upper

front surface expression only

$$\iint \delta u \, G \frac{\partial u}{\partial z} \, dx \, dy = \iint \frac{p}{(1+u_x)^2} \frac{\partial u}{\partial z} \, \delta u \, dx \, dy$$

remains and for any chosen δu,

$$\frac{\partial u}{\partial z} = 0 .$$

The same applies to the lower front surface

$$\frac{\partial v}{\partial z} = 0 .$$

With the condition w = 0 we have for the limiting conditions at the front surfaces

$$w = 0, \ \frac{\partial u}{\partial z} = 0, \ \frac{\partial v}{\partial z} = 0 \quad \text{oder} \quad w = 0, \ \tau_{zx} = 0, \ \tau_{zy} = 0 .$$

front surfaces

Finally we have that the "most dangerous displacement" u,v,w is sufficient for the homogeneous differential equations

$$G\left(\Delta u+\frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right)=\frac{\rho}{(1+\alpha_x)^2}\frac{\partial^2 u}{\partial z^2},$$
$$G\left(\Delta v+\frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right)=\frac{\rho}{(1+\alpha_y)^2}\frac{\partial^2 v}{\partial z^2},$$
$$G\left(\Delta w+\frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right)=\frac{\rho}{(1+\alpha_z)^2}\frac{\partial^2 w}{\partial z^2}$$

with homogeneous limiting or boundary conditions:

a) for the shell

$$\tau_{xy}\cos(n,y)+\sigma_x\cos(n,x)=0,$$
$$\sigma_y\cos(n,y)+\tau_{xy}\cos(n,x)=0,$$
$$\tau_{yz}\cos(n,y)+\tau_{xz}\cos(n,x)=0,$$

b) for the front surfaces

$$w=0, \qquad u_z=0, \qquad v_z=0,$$

and the tensions according to the classical theory

$$\sigma_x=2G\left(\frac{\partial u}{\partial x}+\frac{\Theta}{m-2}\right), \qquad \tau_{xy}=G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

Equations (13) give a simple meaning which gives the connection of the extended theory with the elementary approximation theory. When in the equilibrium the equations of the classical elasticity theory

$$G\left(\Delta u+\frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right)+X=0,$$
$$G\left(\Delta v+\frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right)+Y=0,$$
$$G\left(\Delta w+\frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right)+Z=0$$

and

$$X = -\frac{p}{(1+\alpha_x)^2} u_{zz}, \quad Y = \frac{p}{(1+\alpha_y)^2} v_{zz}, \quad Z = -\frac{p}{(1+\alpha_z)^2} w_{zz}$$

are put as the volume forces we obtain equations (13). This reveals that the integration of these equations is identical to the task dealing with finding the equilibrium profile of the rod when, at a proper p-value, the loads per each volume unit are proportional to the magnitudes u_{zz}, v_{zz}, w_{zz} .

When the volume load over the cross-section of the rod is integrated we obtain loads per length unit of ~~hte~~ the rod

$$\int X\,dF = -\frac{P}{(1+\alpha_x)^2}\frac{d^2[u]}{dz^2}, \quad \int Y\,dF = -\frac{P}{(1+\alpha_y)^2}\frac{d^2[v]}{dz^2}, \quad \int Z\,dF = -\frac{P}{(1+\alpha_z)^2}\frac{d^2[w]}{dz^2},$$

where $\frac{d^2[u]}{dz^2}$ etc. are the average values. If these average values are replaced by the differential ratios $\frac{d^2 u}{dz^2}, \frac{d^2 v}{dz^2}$ taken for the rod's centreline, and the small magnitude α in the denominator is neglected, the task arises to find the equilibrium profile of the rod which is loaded vertically with forces $-P\frac{d^2 u}{dz^2}, -P\frac{d^2 v}{dz^2}$. This is exactly the content of equation $EIw^{IV} = -Pw^{II}$ which follows from the elementary theory.

III. Numerical Results

The Jacobi's equations together with the boundary conditions of page 361 (23) tell us that there exists a neighboring equilibrium state next to the initial conditions. The displacements that transfer the initial conditions into the neighboring conditions are the similar factors $(1+\alpha_x)^2$ etc. solutions u,v,w. Since the Jacobi's equations and the boundary conditions are homogeneous we have a proper value problem. The proper value is the"critical" load p.

In this section we solve the proper value problem for circular and rectangular cross-sections. In the first case - by integrating by development in series; in the second by the Ritz method.

10 Circular cross-section

Equation (13)

$$G\left(\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right) = p\frac{\partial^2 u}{\partial z^2},$$
$$G\left(\Delta v + \frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right) = p\frac{\partial^2 v}{\partial z^2}, \quad \Theta = u_x + v_y + w_z$$
$$G\left(\Delta w + \frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right) = p\frac{\partial^2 w}{\partial z^2}$$

is transformed for cylindrical coordinates. This means we seek the equilibrium profile of a rod loaded per volume unit by forces:

$$-p\frac{\partial^2 u}{\partial z^2} = X, \quad -p\frac{\partial^2 v}{\partial z^2} = Y, \quad -p\frac{\partial^2 w}{\partial z^2} = Z$$

where the very small magnitudes a_x, a_y, a_z are negligible when compared to 1.

When the cylindrical coordinates are introduced r, ϑ, z and the displacements are: radial ϱ, tangential τ and w axial, the problem is to find the equilibrium profile for a rod loaded in the three directions by volume forces $-p\frac{\partial^2 \varrho}{\partial z^2}, -p\frac{\partial^2 \tau}{\partial z^2}, -p\frac{\partial^2 w}{\partial z^2}$

when $\sigma_r, \sigma_\vartheta, \sigma_z, \tau_{r\vartheta}, \tau_{\vartheta z}, \tau_{rz}$

are the cylindrical coordinates of the tension forces.

The equilibrium conditions [5] are

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\vartheta}}{\partial \vartheta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\vartheta}{r}=p\frac{\partial^2 \varrho}{\partial z^2},$$

$$\frac{\partial \tau_{r\vartheta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\vartheta}{\partial \vartheta}+\frac{\partial \tau_{\vartheta z}}{\partial z}+\frac{2\tau_{r\vartheta}}{r}=p\frac{\partial^2 \tau}{\partial z^2},$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\vartheta z}}{\partial \vartheta}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=p\frac{\partial^2 w}{\partial z^2}$$

and the tension-eapansion equations are

$$\sigma_r=2G\left(\frac{\partial \varrho}{\partial r}+\frac{\Theta}{m-2}\right),\qquad \tau_{\vartheta z}=G\left(\frac{1}{r}\frac{\partial w}{\partial \vartheta}+\frac{\partial \tau}{\partial z}\right),$$

$$\sigma_\vartheta=2G\left(\frac{1}{r}\frac{\partial \tau}{\partial \vartheta}+\frac{\varrho}{r}+\frac{\Theta}{m-2}\right),\qquad \tau_{zr}=G\left(\frac{\partial \varrho}{\partial z}+\frac{\partial w}{\partial r}\right),$$

$$\sigma_z=2G\left(\frac{\partial w}{\partial z}+\frac{\Theta}{m-2}\right),\qquad \tau_{\vartheta r}=G\left(\frac{\partial \tau}{\partial r}-\frac{\tau}{r}+\frac{1}{r}\frac{\partial \varrho}{\partial z}\right),$$

$$\Theta=\frac{\partial \varrho}{\partial r}+\frac{1}{r}\frac{\partial \tau}{\partial \vartheta}+\frac{\varrho}{r}+\frac{\partial w}{\partial z}.$$

By eliminating tensions in the equilibrium conditions through a tension expansion equation we have Jacobi's equations with cylindrical coordinates

$$\left.\begin{aligned}
&\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\\
&G\left[\Delta\varrho+\frac{m}{m-2}\frac{\partial \Theta}{\partial r}-\frac{2}{r^2}\frac{\partial \tau}{\partial \vartheta}-\frac{\varrho}{r^2}\right]=p\frac{\partial^2 \varrho}{\partial z^2},\\
&G\left[\Delta\tau+\frac{m}{m-2}\frac{1}{r}\frac{\partial \Theta}{\partial r}+\frac{2}{r^2}\frac{\partial \varrho}{\partial \vartheta}-\frac{\tau}{r^2}\right]=p\frac{\partial^2 \tau}{\partial z^2},\\
&G\left[\Delta w+\frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right]=p\frac{\partial^2 w}{\partial z^2}
\end{aligned}\right\}\quad\ldots\ldots\ldots\quad (14).$$

Their boundary conditions

a) $\sigma_r=0,\quad \tau_{r\vartheta}=0,\quad \tau_{rz}=0$ for the cylindrical shell

and b) $w=0,\quad u_z=0,\quad r_z=0$ for the front planes $(z=0,\ z=L)$.

(z=0, z=L)

To integrate (14) we chose

$$
\left.\begin{aligned}
\varrho &= P(r)\cos n\vartheta \cos \nu z,\\
\tau &= Q(r)\sin n\vartheta \cos \nu z,\\
w &= R(r)\cos n\vartheta \sin \nu z
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (15)
$$

It is sufficient for boundary conditions at the front planes when we take

$$
\nu = \frac{2\pi}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)
$$

For functions P(r), Q(r), R(r) three common differential equations follow from (14)

$$
\left.\begin{aligned}
&P'' + \frac{1}{r}P' - \left|1 + \frac{m-2}{2(m-1)}n^2 - \nu^2(q-1)\frac{m-2}{2(m-1)}r^2\right|\frac{P}{r^2}\\
&\qquad - \frac{3m-4}{2(m-1)}n\frac{Q}{r^2} + \frac{mn}{2(m-1)}\frac{Q'}{r} + \frac{m\nu}{2(m-1)}R' = 0,\\
&Q'' + \frac{1}{r}Q' - \left|1 + \frac{2(m-1)}{m-2}n^2 - \nu^2(q-1)r^2\right|\frac{Q}{r^2}\\
&\qquad - \frac{3m-4}{m-2}n\frac{P}{r^2} - \frac{mn}{m-2}\frac{P'}{r} - \frac{mn\nu}{m-2}\frac{R}{r} = 0,\\
&R'' + \frac{1}{r}R' + \nu^2\left|q - \frac{2(m-1)}{m-2} - \frac{n^2}{r^2}\right|R\\
&\qquad - \frac{m\nu}{m-2}P' - \frac{m\nu}{m-2}\frac{P}{r} - \frac{mn\nu}{m-2}\frac{Q}{r} = 0
\end{aligned}\right\} \quad \ldots\ldots \quad (17)
$$

Where the vertical lines signify derivatives according to

and q equals p/G.

Through the inotroduction of cylindrical coordinates it was succeeded to reduce the partial system (14) to a system with three common differential equations of second order.

Equations (17) supply the case of a break, that is a linear bending of a solid cylinder and a hollow wall column for n=1. The equation s are, when the cross-contraction value is 10/3:

$$\left.\begin{aligned} & r^2 P'' + r P' + \frac{2}{7} \nu^2 (q-1) r^2 P - \frac{9}{7} P - \frac{9}{7} Q + \frac{5}{7} r Q' + \frac{5\nu}{7} r^2 R' = 0, \\ & r^2 Q'' + r Q' + \nu^2 (q-1) r^2 Q - \frac{9}{2} Q - \frac{9}{2} P - \frac{5}{2} r P' - \frac{5\nu}{2} r R = 0, \\ & r^2 R'' + r R' + \nu^2 \left(q - \frac{7}{2}\right) R - R - \frac{5\nu}{2} r P - \frac{5\nu}{2} r^2 P' - \frac{5\nu}{2} r Q = 0 \end{aligned}\right\} \quad \ldots \quad (18).$$

The boundary conditions for the shell are simplified by using (15) in equations

$$\left.\begin{aligned} & 7 r_a P' + 3(P+Q) + 3 \nu r_a R = 0, \\ & P + Q - r_a Q' = 0, \\ & R' - \nu P = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (19).$$

and in case of a hollow cylinder there are three analogous equations with r_i instead of r_a .

A complete integration of (18) leads to six integration constants. The correspond with six boundary conditions (three at the inside and three at the outside boundaries). For the solid cylinder there are three conditions for the outside boundary. For the three missing conditions we have that for tensions $\sigma_r, \tau_{r\vartheta}, \tau_{rz}$ for r=0 are finite. Functions P,Q,R for $r=0$ must

behave in a regular manner. This regularity requirement supplies three more data for the determination of constants.

Equations (18) are integrated normally by developing exponential series for the solid cylinder. The succesive calculations of coefficients delivers, designating the coefficients c_1, c_2, c_3, the series:

$$
\left.\begin{aligned}
P(r) = \frac{c_1}{\nu}\Big| 1 &+ \frac{147-22q}{7\cdot 192}(q-1)\nu^4 r^4 + \frac{434-343q+34q^2}{7\cdot 96\cdot 96}(q-1)\nu^6 r^6 + \ldots \Big| \\
+ \nu c_2 \Big| & r^2 + \frac{154-29q}{7\cdot 24}\nu^2 r^4 + \frac{441-257q+41q^2}{7\cdot 12\cdot 96}\nu^4 r^6 + \ldots \Big| \\
+ c_3 \Big| & \frac{140-15q}{7\cdot 96}\nu^3 r^4 + \frac{420-315q+20q^2}{7\cdot 48\cdot 96}\nu^5 r^6 + \ldots \Big|,
\end{aligned}\right\} \quad (20)
$$

$$
\left.\begin{aligned}
-Q(r) = \frac{c_1}{\nu}\Big| & 1 + 2(q-1)\nu^2 r^2 + \frac{231-206q}{7\cdot 192}(q-1)\nu^4 r^4 + \frac{2450-4375q+2050q^2}{49\cdot 64\cdot 144}(q-1)\nu^6 r^6 + \ldots \Big| \\
+ \nu c_2 \Big| & 19 r^2 + \frac{266-241q}{7\cdot 24}\nu^2 r^4 + \frac{2793-5061q+2393q^2}{49\cdot 32\cdot 144}\nu^4 r^6 + \ldots \Big| \\
+ c_3 \Big| & 5\nu r^2 + \frac{280-255q}{7\cdot 96}\nu^3 r^4 + \frac{2940-5355q+2540q^2}{49\cdot 32\cdot 144}\nu^5 r^6 + \ldots \Big|, \\
-R(r) = \frac{c_1}{\nu}\Big| & \frac{5}{8}(q-1)\nu^3 r^3 + \frac{70-45q}{7\cdot 192}(q-1)\nu^5 r^5 + \frac{735-945q+335q^2}{49\cdot 96\cdot 96}(q-1)\nu^7 r^7 + \ldots \Big| \\
+ \nu c_2 \Big| & 5\nu r^3 + \frac{70-45q}{7\cdot 24}\nu^3 r^5 + \frac{735-945q+335q^2}{49\cdot 12\cdot 96}\nu^5 r^7 + \ldots \Big| \\
+ c_3 \Big| & -r + \frac{9+q}{8}\nu^2 r^3 + \frac{133-76q-7q^2}{7\cdot 192}\nu^4 r^5 + \frac{1421-1743q+523q^2+49q^3}{49\cdot 96\cdot 96}\nu^6 r^7 + \ldots \Big|
\end{aligned}\right\} \quad (20).
$$

Apaet from these solutions the system (18) also contains logarhytmically singular solutions which fall out because of the regularity requirement.

Multiplying the series by the trigonometric functions of and z, according to (15), when q is found, we obtain the values for the "most dangerous" displacements from equilibrium for the cylindrical rod. We still have the three arbitrary integration constants c_1, c_2, c_3 which have to be determined from the boundary conditions (19). They are:

$$7 r_a P'' + 3(P+Q) + 3 \nu r_a R = 0,$$
$$P + Q - r_a Q' = 0,$$
$$R' - \nu P = 0$$

and, say, that the shell becomes tension free.

If we introduce P,Q,R values for $r=r_a$ in (20) then the boundary conditions form a system of three homogeneous linear equations for the constants c_1, c_2, c_3. The magnitude $\nu = \frac{2\pi}{L}$ and the radius r_a appear with same exponents so that for their product $k = \frac{2\pi r_a}{L}$ this system may be solved for values different from zero only if the determinant D of the system disappears. Since the determinant contains the unknown load magnitude q, it can be set in such a waybthat the determinant becomes zero.

For the following calculations the series for P,Q,R we used the values given in (20). An ample calculation of the determinant

leads to the equation

$$\left.\begin{array}{l} \frac{40}{3} q k^2 - \frac{546 - 1099 q + 503 q^2}{63} k^4 \\ \quad + \frac{12495 - 36043 q + 31781 q^2 - 8358 q^3}{49 \cdot 144} k^6 - \ldots = 0 \end{array}\right\} \quad \ldots\ldots\ldots (21).$$

Further calculation is possible only numerically. Because of the complex structure of (21) with respect to q, load values are used to solve the series. Since q=p/G and the push module G is very high, q must be small. With G=800,000 kg/cm^2 we obtain for the loads p, which are within Hook's law, p=800kg/cm^2 and p=1,600kg/cm^2 the values q=1/1,000 or q=1/500.

The results of equation (21) were compared with the elementary theory where the break lenghth L_k is given by

$$L_k^2 = 4\pi^2 \frac{EI}{P_k}.$$

The result is that there occurs only a small deviation from Euler's formula. It is

$$q = \frac{1}{1000}, \qquad L^2 = 4\pi^2 \frac{EI}{P_x}(1 - 0{,}0012),$$

$$q = \frac{1}{500}, \qquad L^2 = 4\pi^2 \frac{EI}{P_x}(1 - 0{,}0027).$$

Thus within the calculation accuracy Euler's break formula is confirmed.

11. Rectangular cross-section

The rod has a lenghth L. The rectangle's dimensions are 2a and 2b(a > b). The position and the tension relationships are the same as before.

The stability limit is dominated by equation $\delta Q_1 = \delta Q_2$ where Q_1 and Q_2 have the meaning of page 360 (23).

The Ritz method is used to solve the equation.

For displacements u,v,w we chose

$$u = A(xy)\cos\nu z,\quad v = B(xy)\cos\nu z,\quad w = C(xy)\sin\nu z,\quad \nu = \frac{2\pi}{L} \quad \ldots\ldots \quad (22)$$

they reduce the spatial problem to a plane problem. It satisfies the boundary conditions for the front planes, where for z=0 and z=L it has to be w=0, u_z=0, v_z =0. The integration of Q_1 and Q_2 over the rod lenghth L gives

$$\left.\begin{aligned} Q_1 &= \frac{GL}{2}\int_{-a}^{+a}\int_{-b}^{+b}\Big\{\frac{m-1}{m-2}(A_x + B_y + \nu C)^2 - 2A_x B_y - 2\nu B_y C - 2\nu A_x C \\ &\qquad + \frac{1}{2}(A_y + B_x)^2 + \frac{1}{2}(C_y - \nu B)^2 + \frac{1}{2}(C_x - \nu A)^2\Big\}\,dx\,dy, \\ Q_2 &= \frac{\nu^2 L}{2}\int_{-a}^{+a}\int_{-b}^{+b}[A^2 + B^2 + C^2]\,dx\,dy \end{aligned}\right\} \quad . \quad (23).$$

For A,B,C we chose

$$\left.\begin{aligned} A &= \frac{a_0}{\nu} + a_1 x + a_2 y + a_{11}\nu x^2 + 2a_{12}\nu xy + a_{22}\nu y^2, \\ B &= \frac{\beta_0}{\nu} + \beta_1 x + \beta_2 y + \beta_{11}\nu x^2 + 2\beta_{12}\nu xy + \beta_{22}\nu y^2, \\ C &= \frac{\gamma_0}{\nu} + \gamma_1 x + \gamma_2 y + \gamma_{11}\nu x^2 + 2\gamma_{12}\nu xy + \gamma_{22}\nu y^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (24).$$

Introduced in (23) we have

$$\begin{aligned} Q_1 = 2abLG\Big\{&\Big|\frac{7}{4}(a_1^2 + \beta_2^2 + \gamma_0^2) + \frac{1}{2}(a_0^2 + \beta_0^2 + a_2^2 + \beta_1^2 + \gamma_1^2 + \gamma_2^2) \\ &+ \frac{3}{2}(a_1\beta_2 + a_1\gamma_0 + \beta_2\gamma_0) + a_2\beta_1 - \beta_0\gamma_2 - a_0\gamma_1\Big| \\ &+ \Big|\frac{a_1^2 + \beta_1^2}{2} + 7(a_{11}^2 + \beta_{12}^2) + 2(\beta_{11}^2 + \gamma_{11}^2 + a_{12}^2 + \gamma_{12}^2) + \frac{7}{4}\gamma_1^2 \\ &+ a_0 a_{11} + \beta_0\beta_{11} + \frac{7}{2}\gamma_0\gamma_{11} - \frac{1}{2}a_1\gamma_{11} + \frac{3}{2}\beta_2\gamma_{11} + 2a_{11}\gamma_1 \\ &+ 6a_{11}\beta_{12} + 3\gamma_1\beta_{12} + 1a_{12}\beta_{11} - \beta_{11}\gamma_2 - 2\beta_1\gamma_{12}\Big|\frac{\nu^2 a^2}{3} \\ &+ \Big|\frac{a_2^2 + \beta_2^2}{2} + 7(a_{12}^2 + \beta_{22}^2) + 2(a_{22}^2 + \gamma_{22}^2 + \beta_{12}^2 + \gamma_{12}^2) + \frac{7}{4}\gamma_2^2 \\ &+ a_0 a_{22} + \beta_0\beta_{22} + \frac{7}{2}\gamma_0\gamma_{22} - \frac{1}{2}\beta_2\gamma_{22} + \frac{3}{2}a_1\gamma_{22} + 2\beta_{22}\gamma_2 \end{aligned}$$

$$+6\,\alpha_{12}\,\beta_{22}+3\,\gamma_2\,\alpha_{12}+4\,\alpha_{22}\,\beta_{12}-\alpha_{22}\,\gamma_1-2\,\alpha_0\,\gamma_{12}\Big|\frac{\nu^2 b^2}{3}$$

$$+\Big|2\,(\alpha_{12}^{\,2}+\beta_{12}^{\,2})+\alpha_{11}\,\alpha_{22}+\beta_{11}\,\beta_{22}+\frac{7}{2}\,(\gamma_{11}\gamma_{22}+2\,\gamma_{12}^{\,2})\Big|\frac{\nu^4 a^2 b^2}{9}$$

$$+\Big|\frac{\alpha_{11}^{\,2}+\beta_{11}^{\,2}}{2}+\frac{7}{4}\,\gamma_{11}^{\,2}\Big|\frac{\nu^4 a^4}{5}+\Big|\frac{\alpha_{22}^{\,2}+\beta_{22}^{\,2}}{2}+\frac{7}{4}\,\gamma_{22}^{\,2}\Big|\frac{\nu^4 b^4}{5}\Big\};$$

$$Q_2=a\,b\,L\,p\Big\{\Big|\alpha_0^{\,2}+\beta_0^{\,2}+\gamma_0^{\,2}\Big|$$

$$+\Big|\alpha_1^{\,2}+\beta_1^{\,2}+\gamma_1^{\,2}+2\,\alpha_0\,\alpha_{11}+2\,\beta_0\,\beta_{11}+2\,\gamma_0\,\gamma_{11}\Big|\frac{\nu^2 a^2}{3}$$

$$+\Big|\alpha_2^{\,2}+\beta_2^{\,2}+\gamma_2^{\,2}+2\,\alpha_0\,\alpha_{22}+2\,\beta_0\,\beta_{22}+2\,\gamma_0\,\gamma_{22}\Big|\frac{\nu^2 b^2}{3}$$

$$+\Big|4\,\alpha_{12}^{\,2}+4\,\beta_{12}^{\,2}+4\,\gamma_{12}^{\,2}+2\,\alpha_{11}\,\alpha_{22}+2\,\beta_{11}\,\beta_{22}+2\,\gamma_{11}\,\gamma_{22}\Big|\frac{\nu^4 a^2 b^2}{9}$$

$$+\Big|\alpha_{11}^{\,2}+\beta_{11}^{\,2}+\gamma_{11}^{\,2}\Big|\frac{\nu^4 a^4}{5}+\Big|\alpha_{22}^{\,2}+\beta_{22}^{\,2}+\gamma_{22}^{\,2}\Big|\frac{\nu^4 b^4}{5}\Big\}.$$

Integrals Q_1 and Q_2 became the quadratic functions of the 18 ~~different~~ coefficients. The equations for the stability limit $\delta Q_1=\delta Q_2$, yields 18 linear equations for the coefficients according to $\frac{\partial Q_1}{\partial \alpha_{ix}}=\frac{\partial Q_2}{\partial \alpha_{ix}}$. The 18 equations system is distributed into 4 groups which contain the coefficients

1. α_2; β_1; γ_{12}
2. α_1; β_2; γ_0; γ_{11}; γ_{22}
3. α_0; α_{11}; α_{22}; β_{12}; γ_1
4. β_0; β_{11}; β_{22}; α_{12}; γ_2.

The individual groups are systems of homogeneous linear equations where coefficients have a non-zero solution only when the determinant disappears. If we take out one group and equate its deerminant to zero, this determines the break load, we will obtain the coefficients of this group up to a common factor.

The coefficients of the other group become zero because the determinants of the other group for the break load value found do not disappear. We are interested here in the determinant equation

which supplies the break load, thus the question arises: which of the four groups should we take? Naturally such for which the lowest break load results; which one it is - is not difficult to recognize. If the rod breaks in the x direction then all the cross-section points have about the same displacement in the x direction; u cannot be thus an uneven function of x. Now the only group which has no uneven function for u is the third group. Putting the determinant of this group as equal to zero we should obtain the break load value which causes a break in the x-direction. The fourth group, which follows from the third group by substituting x and y, yields the break load for a y break. Thendeterminant equation is the same as for the third group when a and b are substituted. Groups 1 and 2, which with the substitution become one, give a kind of squeeze boundary, but this value has no physical meaning because these phenomena take place outside the limits of the expanded Hook's law.

We write the third group of equations as follows

$$18\,\alpha_{11} + 12\,n^2\,\alpha_{22} + [12 + 12\,n^2 + 4\,n^2 k^2 (1-q)]\,\beta_{12} + 9\,\gamma_1 = 0$$

$$3(1-q)\,\alpha_0 + (1-q)\,k^2\,\alpha_{11} + (1-q)\,n^2 k^2\,\alpha_{22} - 3\,\gamma_1 = 0$$

$$15(1-q)\,\alpha_0 + 5(1-q)\,k^2\,\alpha_{11} + [9(1-q)\,n^2 k^2 + 60]\,\alpha_{22} + 60\,\beta_{12} - 15\,\gamma_1 = 0$$

$$15(1-q)\,\alpha_0 + [9(1-q)\,k^2 + 210]\,\alpha_{11} + 5(1-q)\,n^2 k^2\,\alpha_{22} + 90\,\beta_{12} + 30\,\gamma_1 = 0$$

$$-6\,\alpha_0 + 4\,k^2\,\alpha_{11} - 2\,n^2 k^2\,\alpha_{22} + 6\,k^2\,\beta_{12} + [6 + (7 - 2q)\,k^2]\,\gamma_1 = 0$$

And to simplify we take $q = \frac{p}{G}, \quad n = \frac{b}{a}, \quad a\nu = \frac{2\pi a}{L} = k$

The determinant for this system, when multiplied, is

$$
\begin{aligned}
q^5 & - \frac{15n^2k^2 + 9312k^2 + 51}{2n^2k^2} q^4 + \frac{26n^4k^4 + 294n^4k^2 + 432n^2k^2 + 93n^2 + 630}{4n^4k^4} q^3 \\
& - \left| \frac{100n^4k^6 + 2535n^4k^4 + 1296n^2k^4 + 4392n^4k^2 + 22797n^2k^2 + 3660k^2}{4n^4k^6} \right. \\
& \left. + \frac{1890n^4 + 14850n^2 + 28800}{4n^4k^6} \right| q^2 + \left| \frac{60n^4k^8 + 1518n^2k^4 + 1734n^2k^3}{4n^4k^8} \right. \\
& \left. + \frac{4896n^4k^4 + 38421n^2k^4 + 4230k^4 + 1890n^4k^2 + 14850n^2k^2 + 99090k^2 + 81000}{4n^4k^8} \right| q \\
& - \left| \frac{56n^4k^6 + 2568n^4k^4 + 6200n^4k^2 + 1320n^2k^4 + 39735n^2k^2 + 5600k^2 + 280800}{16n^4k^6} \right| = 0
\end{aligned}
\qquad (25).
$$

Since the coefficients of the exponents of q change with its sign, the equation may only have positive roots.

For a numerical calculation we note that q as well as k are very small. If we consider the limit case of very thin rods (we let k tend to zero) then after multiplying by k^8 we are left with the last members in the linear and absolute members which yield the equation for q

$$13k^2 - 15q = 0$$

From this - the break tension

$$p_k = (13/15) G k^2.$$

Introducing

$$G = \frac{m}{2(m+1)} E, \quad m = \frac{10}{3}, \quad k = \frac{2\pi a}{L}, \quad \frac{I}{F} = \frac{a^2}{3}, \quad P_k = p_k \cdot F,$$

we obtain exactly Euler's formula for the tensioned rod:

$$P_k = \frac{4\pi^2 E I}{l^2}.$$

This formula is thus confirmed for the limit case of very thin rods. The lateral relationship n does not appear explicitly. Strictly speaking there is a need for a n-depending correction but in practically important cases this correction does not reach any noticeable effect.

Summary

As an example of the stability theory for elastic equilibrium developed by E.Trefftz the present work calculates the break limit for circular and rectangular rods. The first part outlines the elasticity theory for finite deformations. In particular, it is shown that the expression f r the elastic potential (internal energy of a volume unit) may be taken from the elasticity theory of small deformations when, instead of linearized distortion magnitudes of the classical theory the real distortion magnitudes for finite deformations are taken. (because of this the coefficients of the linear elements change)

From the obtained integraal for the overall inner elastic energy this work develops an expression for the second variation of internal energy, whose sign decides the stability. Then following common methods for variations' calculation, the stability limit is determined, that is the load limit above which the second variation is capable to accept negative values. The third section contains results of numerical calculations from which it follows that within the limits of calculation accuracy (up to fractions of one percent) Euler's formula is

confirmed, which formula is obtained according to elementary beam-bending theory.

I wish to thank here Proffessor Dr. Trefftz for his encouragement in my work.

References and notes

1) Für ältere Arbeiten: Enzyklopädie der math. Wissensch. Band IV. Leipzig 1907/08. Für neuere Arbeiten: Handb. d. Physik v. Geiger und Scheel. Band VI. Leipzig 1928.

2) E. Trefftz, Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichts aus der Theorie endlicher Deformationen. Internationaler Kongress für Technische Mechanik 1930, Stockholm. Teil III. S. 44.

3) M. Lagally, Vorlesungen über Vektorrechnung. Leipzig 1928.

4) This expression was first used by Trefftz in 1930 and was given to me privately.

5) Vgl. Handbuch der Physik. Band VI. S. 41.

DISTRIBUTION LIST

DISTRIBUTION DIRECT TO RECIPIENT

ORGANIZATION	MICROFICHE	ORGANIZATION	MICROFICHE
A205 DMATC	1	E053 AF/INAKA	1
A210 DMAAC	2	E017 AF/ RDXTR-W	1
B344 DIA/RDS-3C	8	E404 AEDC	1
C043 USAMIIA	1	E408 AFWL	1
C509 BALLISTIC RES LABS	1	E410 ADTC	1
C510 AIR MOBILITY R&D LAB/FIO	1	E413 ESD	2
		FTD	
C513 PICATINNY ARSENAL	1	CCN	1
C535 AVIATION SYS COMD	1	ETID	3
		NIA/PHS	1
C591 FSTC	5	NICD	5
C619 MIA REDSTONE	1		
D008 NISC	1		
H300 USAICE (USAREUR)	1		
P005 ERDA	1		
P055 CIA/CRS/ADD/SD	1		
NAVORDSTA (50L)	1		
NASA/KSI	1		
AFIT/LD	1		